On the Transformation of Elliptic Functions (Sequel).

By Prof. Cayley.

The chief object of the present paper is the further development of the $\rho\alpha\beta$-theory in the case $n = 7$. I recall that the forms are

$$\frac{dy}{\sqrt{1 - 2\beta y^2 + y^4}} = \frac{\rho\,dx}{\sqrt{1 - 2\alpha x^2 + x^4}},$$

where
$$y = \frac{x(\rho + A_2 x^2 + A_1 x^4 + x^6)}{1 + A_1 x^2 + A_2 x^4 + \rho x^6}.$$

The paragraphs are numbered consecutively with those of the former paper "On the Transformation of Elliptic Functions," vol. IX, pp. 193–224.

The Seventhic Transformation: the $\rho\alpha$-Equation.   Art. Nos. 51 to 57.

51. The equation is given incorrectly Nos. 7 and 42; there was an error of sign in a term $512\alpha^3\rho$, which affected also the coefficient of $\alpha\rho$, and an error of sign in the absolute term 7. The correct form is

$$\rho^8 - 28\rho^6 - 112\alpha\rho^5 - 210\rho^4 - 224\alpha\rho^3 + (-1484 + 1344\alpha^2)\rho^2$$
$$+ (464\alpha - 512\alpha^3)\rho - 7 = 0;$$

or, arranging in powers of α, this is

$$\alpha^3 . 512\rho$$
$$+ \alpha^2 . - 1344\rho^2$$
$$+ \alpha . 112\rho^5 + 224\rho^3 - 464\rho$$
$$- (\rho^8 - 28\rho^6 - 210\rho^4 - 1484\rho^2 - 7) = 0.$$

This may also be written in the forms

$$(\alpha - 1)\{\alpha^2 . 512\rho + \alpha(-1344\rho^2 + 512\rho) + 112\rho^5 + 224\rho^3 - 1344\rho^2 + 48\rho\}$$
$$- (\rho + 1)^7(\rho - 7) = 0,$$

and

$$(\alpha + 1)\{\alpha^2 . 512\rho + \alpha(-1344\rho^2 - 512\rho) + 112\rho^5 + 224\rho^3 + 1344\rho^2 + 48\rho\}$$
$$- (\rho - 1)^7(\rho + 7) = 0.$$

To simplify the $\rho\alpha$-equation we assume $A = 8\rho\alpha - 7\rho^2$; then the $A\rho$-equation is

$$A^3$$
$$+ A\rho^2 (14\rho^4 - 119\rho^2 - 58)$$
$$- \rho^2 (\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7) = 0;$$

viz., this is a cubic equation wanting its second term, and so at once solvable by Cardan's formula: say the equation is

$$A^3 + A\rho^2 q_1 - \rho^2 r_1 = 0,$$

where

$$q_1 = 14\rho^4 - 119\rho^2 - 58,$$
$$r_1 = \rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7.$$

It is convenient to recall here that, writing $\sigma = -\dfrac{7}{\rho}$, and $B = 8\sigma\beta - 7\sigma^2$, we have between σ, β, B precisely the same equations as between ρ, α, A; $\rho = 1$ gives $\sigma = -7$, and we have as corresponding values $\alpha = -1$, $A = -15$, $\beta = -1$, $B = -287$: these are very convenient for verification of the formulæ. Similarly $\rho = -7$ gives $\sigma = 1$, and then $\alpha = -1$, $A = -287$, $\beta = -1$, $B = -15$; but I have in general used the former values only.

52. We have $\qquad A = f + g,$

where

$$3fg = -\rho^2 q_1,$$
$$f^3 + g^3 = \rho^2 r_1,$$

and thence

$$f^3 - g^3 = \rho^2 \sqrt{r_1^2 + \frac{4\rho^2 q_1^3}{27}}.$$

We have identically

$$27 (\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7)^2 + 4\rho^2 (14\rho^4 - 119\rho^2 - 58)^3$$
$$= (\rho^6 + 75\rho^4 - 141\rho^2 + 1)^2 (27\rho^4 + 122\rho^2 + 1323)$$

[$\rho = 1$, this is $27.930^2 + 4(-163)^3 = 64^2.1472$; that is, $23352300 - 17322988 = 6029312$, which is right]; but it is convenient to divide by 27, so as instead of $27\rho^4 + 122\rho^2 + 1323$ to have in the formulæ $\rho^4 + \dfrac{122}{27}\rho^2 + 49$, or say

$$\rho^4 + K\rho^2 + 49 \left(K = \frac{122}{27}\right).$$

Hence writing

$$t_1 = \rho^6 + 75\rho^4 - 141\rho^2 + 1,$$
$$\delta = \rho^4 + K\rho^2 + 49,$$

we have

$$r_1^2 + \frac{4}{27}\rho^2 q_1 = t_1^2 \delta,$$

and consequently
$$2f^3 = \rho^2 (r_1 + t_1 \sqrt{\delta}),$$
$$2g^3 = \rho^2 (r_1 - t_1 \sqrt{\delta}).$$

53. It was easy to foresee that the cube root of $r_1 \pm t_1 \sqrt{\delta}$ would break up into the form $(U \pm \sqrt{\delta})\sqrt[3]{W \pm \sqrt{\delta}}$, and I was led to the actual expressions by the identities

$$20(14\rho^4 - 119\rho^2 - 58) = (19\rho^2 - 53)^2 - 3(27\rho^4 + 122\rho^2 + 1323);$$

that is, $20q_1 \qquad\qquad = (19\rho^2 - 53)^2 - 81\delta,$

and
$$27(\rho^2 - 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = -500\rho^2,$$
$$27(\rho^2 + 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = 256\rho^2;$$

or, as these may be written,
$$(\rho^2 - 7)^2 - \delta = -\frac{500}{27}\rho^2, \quad (\rho^2 + 7)^2 - \delta = \frac{256}{27}\rho^2.$$

We in fact have further the two identities

$$1000(\rho^6 + 75\rho^4 - 141\rho^2 + 1)$$
$$= \{(19\rho^2 - 53)^3 + 243(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\}$$
$$+ \{27(19\rho^2 - 53)^2 \qquad + 729(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7),$$

$$-1000(\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7)$$
$$= \{(19\rho^2 - 53)^3 + 243(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7)$$
$$+ \{27(19\rho^2 - 53)^2 \qquad + 729(\rho^4 + K\rho^2 + 49)\}(\rho^4 + K\rho^2 + 49),$$

viz., writing $\qquad 19\rho^2 - 53 = 9U, \quad -\rho^2 + 7 = W,$
these equations become

$$\frac{1000}{729} t_1 = U^3 + 3U\delta + (3U^2 + \delta)W,$$

$$-\frac{1000}{729} r_1 = (U^3 + 3U\delta)W + (3U^2 + \delta)\delta,$$

and we have thus
$$-\frac{1000}{729}(r_1 - t_1\sqrt{\delta}) = (U + \sqrt{\delta})^3(W + \sqrt{\delta}),$$

and the like equation with $-\sqrt{\delta}$ in place of $\sqrt{\delta}$.

54. In part verification of the last-mentioned identities, observe that in the first of them, putting $\rho = 1$, and comparing first the coefficients of ρ^6 and then the coefficients of ρ^0, we ought to have

$$1000 = 19^3 + 243.19 - (27.19^2 + 729), = 11476 - 10476,$$
$$1000 = (-53^3 - 243.53.49) + (27.53^2 + 729.49)\,7, = -779948 + 780948,$$

which are right; and similarly in the second equation, comparing first the coefficients of ρ^8 and next those of ρ^0, we have

$$- 1000 = (19^3 + 243.19)(- 1) + (27.19^2 + 729), = - 11476 + 10476,$$
$$+ 7000 = (- 53^3 - 243.53.49)(7) + (27.53^2 + 729.49) 49,$$
$$= - 5459636 + 5466636,$$

which are right.

55. We have now $A = f + g$, where

$$f = - \frac{9}{10} (U - \sqrt{\delta}) \sqrt[3]{\frac{1}{2} \rho^2 (W - \sqrt{\delta})},$$

$$g = - \frac{9}{10} (U + \sqrt{\delta}) \sqrt[3]{\frac{1}{2} \rho^2 (W + \sqrt{\delta})},$$

(where observe that, multiplying these two values, we have

$$fg = \frac{81}{100} (U^2 - \delta) \sqrt[3]{\frac{1}{4} \rho^4.(W^2 - \delta)}, = \frac{81}{100} (U^2 - \delta) \sqrt[3]{\frac{1}{4} \rho^4. \frac{-500}{27} \rho^2},$$

$$= \frac{81}{100} (U^2 - \delta)\left(- \frac{5}{3} \rho^2\right);$$

that is,

$$fg = - \frac{27}{20} \rho^2 (U^2 - \delta), = - \frac{27}{20} \rho^2. \frac{20q_1}{81} = - \frac{1}{3} \rho^2 q_1,$$

which is right). Or, finally, substituting for U, W, δ their values, we have, for the solution of the $A\rho$-equation, $A = f + g$, where

$$f = - \frac{9}{10} (19\rho^2 - 53 - \sqrt{\rho^4 + K\rho^2 + 49}) \sqrt[3]{\frac{1}{2} \rho^2 \left\{- \rho^2 + 7 - \sqrt{\rho^4 + K\rho^2 + 49}\right\}}, \left(K = \frac{122}{27}\right),$$

$$g = - \frac{9}{10} (19\rho^2 - 53 + \sqrt{\rho^4 + K\rho^2 + 49}) \sqrt[3]{\frac{1}{2} \rho^2 \left\{- \rho^2 + 7 + \sqrt{\rho^4 + K\rho^2 + 49}\right\}}.$$

56. In the case $\rho = 1$, α has a value $= - 1$, giving for A, $= 8\rho\alpha - 7\rho^2$, the value $- 15$; and, in fact, here $\rho^2 = 1$, and the $A\rho$-equation becomes

$$A^3 - 163A + 930 = 0,$$

that is,

$$(A + 15)(A^2 - 15A + 62) = 0,$$

the roots thus being

$$A = - 15, \quad A = \frac{1}{2} (15 \pm i\sqrt{23}).$$

To verify in this case the values given by the solution of the cubic equation, observe that for $\rho^2 = 1$ we have $\delta = 50 + \frac{122}{27}$, $= \frac{1472}{27}$, and therefore

$\sqrt{\delta} = \dfrac{8\sqrt{23}}{3\sqrt{3}}$, $= \dfrac{8\sqrt{69}}{9}$; also, $U = \dfrac{19\rho^2 - 53}{9}$, $= \dfrac{-34}{9}$, and $W = -\rho^2 + 7$, $= 6$.

Hence $U + \sqrt{\delta} = \dfrac{-34 + 8\sqrt{69}}{9}$, and

$$\sqrt[3]{\frac{1}{2}\rho^2(W+\sqrt{\delta})} = \sqrt[3]{3 + \frac{8\sqrt{69}}{9}}, \ = \frac{\sqrt[3]{81 + 12\sqrt{69}}}{3};$$

hence

$$g = -\frac{9}{10}\frac{2(-17+4\sqrt{69})}{9}\frac{1}{3}\sqrt[3]{81+12\sqrt{69}}, \ = \frac{1}{15}(17-4\sqrt{69})\sqrt[3]{81+12\sqrt{69}};$$

but the cube root is $= \dfrac{1}{2}(3 + \sqrt{69})$, and we have $(17 - 4\sqrt{69})(3 + \sqrt{69})$

$= -225 + 5\sqrt{69}$, $= 5(-45 + \sqrt{69})$; that is, $g = \dfrac{1}{6}(-45 + \sqrt{69})$. Simi-

larly $f = \dfrac{1}{6}(-45 - \sqrt{69})$. We have thus the real root $f + g = -15$, and

the imaginary roots $f\omega + g\omega^2$ or $f\omega^2 + g\omega$, $= -\dfrac{15}{2}(\omega + \omega^2) + \dfrac{1}{6}\sqrt{69}(\omega - \omega^2)$,

viz., the first term is $= \dfrac{15}{2}$ and the second is $\pm \dfrac{1}{6}\sqrt{69}.i\sqrt{3}$, $= \pm \dfrac{1}{2}i\sqrt{23}$;

thus the roots are $\dfrac{1}{2}(15 \pm i\sqrt{23})$, as they should be.

57. I found, by considerations arising out of the new theory Nos. 72 *et seq.*, that writing for shortness $m = i\sqrt{3}$, then, for $\rho = m - 2$, the $\rho\alpha$-equation has a root $\alpha = m$; the corresponding values of $A_1\rho^2$ thus are $A = 12m - 31$, $\rho^2 = -4m + 1$, viz., substituting this value for ρ^2 in the $A\rho$-equation, there should be a root $A = 12m - 31$. The equation becomes

$$A^3 + A(3704m - 7653) + 148306m + 206162 = 0,$$

or, as this may be written,

$$(A - 12m + 31)\{A^2 + A(12m - 31) + 2960m + 4062\} = 0,$$

and the roots thus are

$$A = \ \ 12m - 31,$$
$$A = -6m + \frac{31}{2} \pm \frac{1}{2}\sqrt{-12584m - 16777},$$

where the square root is not expressible as a rational function of m.

Expression of β as a Rational Function of α, ρ.   Art. Nos. 58 to 66.

58. Writing $\sigma = -\dfrac{7}{\rho}$, we have β the same function of σ that ρ is of α; hence if $B = 8\sigma\beta - 7\sigma^2$, the $B\sigma$-equation is

$$B^3$$
$$+ B\sigma^2 (14\sigma^4 - 119\sigma^2 - 58)$$
$$- \sigma^2 (\sigma^8 - 126\sigma^6 + 280\sigma^4 - 1078\sigma^2 - 7) = 0,$$

and the expression for B in terms of σ is obtained from that of A by the mere change of ρ into σ. Say we have $B = f' + g'$ where

$$f' = -\frac{9}{10}(U' - \sqrt{\delta'})\sqrt[3]{\frac{1}{2}\sigma^2(W' - \sqrt{\delta'})},$$
$$g' = -\frac{9}{10}(U' + \sqrt{\delta'})\sqrt[3]{\frac{1}{2}\sigma^2(W' + \sqrt{\delta'})};$$

then we have

$$\frac{1}{2}\sigma^2(W' + \sqrt{\delta'}) = \frac{1}{2}\frac{49}{\rho^2}\left(-\frac{49}{\rho^2} + 7 + \sqrt{\frac{2401}{\rho^4} + \frac{49K}{\rho^2} + 49}\right)$$
$$= -\frac{1}{2} \cdot \frac{343}{\rho^4}(-\rho^2 + 7 - \sqrt{\rho^4 + K\rho^2 + 49})$$
$$= -\frac{343}{\rho^6} \cdot \frac{1}{2}\rho^2(W - \sqrt{\delta}),$$

or say

$$\sqrt[3]{\frac{1}{2}\sigma^2(W' + \sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\frac{1}{2}\rho^2(W - \sqrt{\delta})};$$

and similarly

$$\sqrt[3]{\frac{1}{2}\sigma^2(W' + \sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\frac{1}{2}\rho^2(W + \sqrt{\delta})}.$$

The cube roots which enter into the expression of B are thus identical with those in the expression of A, and it hence appears that B can be expressed rationally in terms of A, ρ; or, what is the same thing, β can be expressed rationally in terms of α, ρ.

59. The *à priori* reason is obvious: the $\rho\alpha$-equation is a cubic in α, but of the order 8 in ρ; hence to a given value of α there correspond 8 values of ρ. Similarly the $\sigma\beta$-equation is a cubic in β, but of the order 8 in σ, or if for σ we substitute its value $= -\dfrac{7}{\rho}$, then we have a $\rho\beta$-equation which is a cubic in β, but of the order 8 in ρ. In the absence of any special relation between this $\rho\beta$-equation and the $\rho\alpha$-equation, there would correspond to each of the 8 values of ρ, 3 values of β; that is, to a given value of α there would correspond 8×3, $= 24$ values of β. But, in fact, to a given value of α there correspond

only 8 values of β, and the two cubic equations are related to each other in such wise that this is so; viz., the relation between them is such that it is possible by means of them to express β as a rational function of ρ, α.

60. Returning to the investigation, we have

$$9\,U' = 19\sigma^2 - 53, \; = \frac{19.49}{\rho^2} - 53;$$

or, writing $\;63\,\overline{U} = 53\rho^2 - 931,$

this is $\qquad U' = -\dfrac{7}{\rho^2}\,\overline{U}, \;\text{whence}\; U' \pm \sqrt{\delta'} = -\dfrac{7}{\rho^2}\left(\overline{U} \mp \sqrt{\delta}\right).$

Hence writing

$$\theta = \sqrt[3]{\frac{1}{2}\,\rho^2\,(W - \sqrt{\delta})}, \quad \phi = \sqrt[3]{\frac{1}{2}\,\rho^2\,(W + \sqrt{\delta})},$$

we have $\qquad f = -\dfrac{9}{10}\,(U - \sqrt{\delta})\,\theta, \; f' = -\dfrac{9}{10}\dfrac{49}{\rho^4}\,(\overline{U} + \sqrt{\delta})\,\phi,$

$$g = -\frac{9}{10}\,(U + \sqrt{\delta})\,\phi, \; g' = -\frac{9}{10}\frac{49}{\rho^4}\,(\overline{U} - \sqrt{\delta})\,\theta,$$

so that, putting for shortness

$$L = -\frac{9}{10}\,(U - \sqrt{\delta}), \; \overline{L} = -\frac{9}{10}\frac{49}{\rho^4}\,(\overline{U} - \sqrt{\delta}),$$

$$M = -\frac{9}{10}\,(U + \sqrt{\delta}), \; \overline{M} = -\frac{9}{10}\frac{49}{\rho^4}\,(\overline{U} + \sqrt{\delta}),$$

we have $\qquad A = L\theta + M\phi, \; B = \overline{L}\theta + \overline{M}\phi,$

where θ^3, ϕ^3 and $\theta\phi$ are each of them free from any cube root; we have, in fact,

$$\theta\phi = \sqrt[3]{\frac{1}{4}\,\rho^4\,(W^2 - \delta)}, \; = \sqrt[3]{\frac{1}{4}\,\rho^4 \cdot \frac{-500}{27}\,\rho^2}, \; = -\frac{5}{3}\,\rho^2,$$

and it may be added that

$$3LM\theta\phi = -\rho^2 q_1, \;\text{whence}\; LM = \frac{1}{5}\,q_1,$$

$$L^3\theta^3 + M^3\phi^3 = \rho^2 r_1,$$

$$L^3\theta^3 - M^3\phi^3 = \rho^2 t_1 \sqrt{\delta};$$

these are, in fact, only the equations obtained by writing $L\theta$, $M\phi$ in place of f, g respectively.

61. In the case $\rho = 1$ we have $\sigma = -7$, the equation for B becomes

$$B^3 + 1358525B + 413536578 \qquad = 0;$$

that is, $\qquad (B + 287)(B^2 - 287B + 1440894) = 0,$

and the roots are

$$-287 \;\text{and}\; \frac{1}{2}\left(287 \pm 497i\sqrt{23}\right), \;\text{or, say}\; -7.41 \;\text{and}\; \frac{7}{2}\left(41 \pm 71i\sqrt{23}\right).$$

We have as before, $\sqrt{\delta} = \dfrac{8\sqrt{69}}{9}$, and $\sqrt[3]{\dfrac{1}{2}\,\rho^2(W + \sqrt{\delta})} = \dfrac{1}{3}\sqrt[3]{81 + 12\sqrt{69}} = \theta$;

also, $\overline{U} = \dfrac{-878}{63}$, whence $\overline{U} + \sqrt{\delta} = \dfrac{2(-439 + 28\sqrt{69})}{63}$.   We thus have

$$f' = -\frac{9}{10}\;49\,.\,\frac{2(-439 + 28\sqrt{69})}{63}\,\frac{1}{3}\sqrt[3]{81 + 12\sqrt{69}},$$

$$= -\frac{7}{15}(-439 + 28\sqrt{69})\sqrt[3]{81 + 12\sqrt{69}},$$

or, putting for the cube root its value $= \dfrac{1}{2}(3 + \sqrt{69})$, this is

$$f' = -\frac{7}{30}(-439 + 28\sqrt{69})(3 + \sqrt{69}),\; = -\frac{287}{2} + \frac{497}{6}\sqrt{69}.$$

Similarly $g' = -\dfrac{287}{2} - \dfrac{497}{6}\sqrt{69}$; and forming the values $f' + g'$, $\omega f' + \omega^2 g'$, $\omega^2 f' + \omega g'$, we have the real root -287 and the imaginary roots $\dfrac{1}{2}(287 \pm 497 i\sqrt{23})$, as above.

62. We have the equations

$$B = \overline{L}\theta \quad\;\; + \overline{M}\phi,$$
$$A = L\theta \quad\;\; + M\phi,$$
$$A^2 - 2LM\theta\phi = \frac{M^2\phi^3}{\theta\phi}\,\theta + \frac{L^2\theta^3}{\theta\phi}\,\phi,$$

from which, eliminating θ, ϕ so far as they present themselves linearly on the right-hand side, and in the resulting equation replacing $\theta\phi$ and $LM\theta\phi$ by their values, we have

$$\begin{vmatrix} B, & \overline{L}, & \overline{M} \\ A, & L, & M \\ -\dfrac{5}{3}\,\rho^2\left(A^2 + \dfrac{2}{3}\,\rho^2 q_1\right), & M^2\phi^3, & L^2\theta^3 \end{vmatrix} = 0;$$

that is,

$$B\,(L^3\theta^3 - M^3\phi^3) = A\,(L^2\overline{L}\theta^3 - M^2\overline{M}\phi^3) - \frac{5}{3}\,\rho^2\left(A^2 + \frac{2}{3}\,\rho^2 q_1\right)(L\overline{M} - \overline{L}M).$$

This may be written

$$B\rho^2 t_1\sqrt{\delta} = A\left\{-\frac{729}{1000}\frac{49}{\rho^4}\left[(U - \sqrt{\delta})^2(\overline{U} - \sqrt{\delta})\frac{1}{2}\,\rho^2(W - \sqrt{\delta})\right.\right.$$
$$\left.\left. - (U + \sqrt{\delta})^2(\overline{U} + \sqrt{\delta})\frac{1}{2}\,\rho^2(W + \sqrt{\delta})\right]\right\}$$
$$- \frac{5}{3}\,\rho^2\left(A^2 + \frac{2}{3}\,\rho^2 q_1\right)\frac{81}{100}\frac{49}{\rho^4}\left[(U - \sqrt{\delta})(\overline{U} + \sqrt{\delta})\right.$$
$$\left. - (U + \sqrt{\delta})(\overline{U} - \sqrt{\delta})\right],$$

where the terms in [ ] contain each of them the factor $\sqrt{\delta}$. Omitting this factor from the equation, and multiplying by ρ^2, we have

$$B\rho^4 t_1 = \frac{81}{100} \, 49 \left\{ \frac{9}{10} A \left[(U^2 + 2U\overline{U} + \delta) W + U^2\overline{U} + (2U + \overline{U})\delta \right] \right.$$
$$\left. - \frac{10}{3} \left(A^2 + \frac{2}{3} \rho^2 q_1 \right) (U - \overline{U}) \right\},$$

which I verify at this stage by writing as before, $\rho = 1$. We have $B = -287$, $A = -15$, $t_1 = -64$, $q_1 = -163$, $W = 6$, $U = -\frac{34}{9}$, $\overline{U} = \frac{-878}{63}$; and, omitting intermediate steps, the equation becomes

$$287.64 = \frac{81.49}{100} \left(\frac{2496000}{567} - \frac{2233600}{567} \right), = \frac{81.49}{100.567} \, 262400, = 18368,$$

which is right.

63. We require the values of $(U^2 + 2U\overline{U} + \delta) W + U^2\overline{U} + (2U + \overline{U})\delta$, and of $U - \overline{U}$: I insert some of the steps of the calculation. We have

$$U^2 + 2U\overline{U} + \delta = \frac{1}{63^2} \left\{ (133\rho^2 - 371)(239\rho^2 - 2233) + 63^2(\rho^4 + 49) + 3.49.122\rho^2 \right\}$$

$$= \frac{1}{63^2} \left\{ 35756\rho^4 - 367724\rho^2 + 1022924 \right\}$$

$$= \frac{4}{567} \left\{ 1277\rho^4 - 13133\rho^2 + 36533 \right\}.$$

Multiplying by W, $= -\rho^2 + 7$, we have

$$(U^2 + 2U\overline{U} + \delta) W = \frac{4}{567} \left\{ -1277\rho^6 + 22072\rho^4 - 128464\rho^2 + 255731 \right\}$$

$$= \frac{4}{5103} \left\{ -11493\rho^6 + 198648\rho^4 - 1156176\rho^2 + 2301579 \right\}$$

$$U^2\overline{U} = \frac{1}{81.63} (19\rho^2 - 53)^2 (53\rho^2 - 931)$$

$$= \frac{1}{5103} \left\{ 19133\rho^6 - 442833\rho^4 + 2023911\rho^2 - 2615179 \right\},$$

$$(2U + \overline{U})\delta = \frac{1}{63.27} (319\rho^2 - 1673)(27\rho^4 + 122\rho^2 + 1323)$$

$$= \frac{1}{1701} \left\{ 8613\rho^6 - 6253\rho^4 + 217931\rho^2 - 221379 \right\}$$

$$= \frac{1}{5103} \left\{ 25839\rho^6 - 18759\rho^4 + 653793\rho^2 - 6640137 \right\},$$

whence

$$U^2\overline{U} + (2U + \overline{U})\delta = \frac{1}{5103} \left\{ 44972\rho^6 - 461592\rho^4 + 2677704\rho^2 - 9255316 \right\}$$

$$= \frac{4}{5103} \left\{ 11243\rho^6 - 115398\rho^4 + 669426\rho^2 - 2313829 \right\}.$$

Hence, adding, we obtain

$$(U^2 + 2U\overline{U} + \delta)\, W + U^2\overline{U} + (2U + \overline{U})\,\delta$$

$$= \frac{4}{5103}\,\{- 250\rho^6 + 83250\rho^4 - 486750\rho^2 - 12250\}$$

$$= \frac{-1000}{5103}\,\{ \qquad \rho^6 - \qquad 333\rho^4 + \qquad 1947\rho^2 + \qquad 49\}:$$

and we have at once

$$U - \overline{U} = \frac{1}{63}\,(80\rho^2 + 560) = \frac{80}{63}\,(\rho^2 + 7).$$

64. We now find

$$B\rho^4 t_1 = - 7A\,(\rho^6 - 333\rho^4 + 1947\rho^2 + 49)$$
$$- 56\,(3A^2 + 2\rho^2 q_1)(\rho^2 + 7),$$

viz. substituting for t_1, q_1 their values, this is

$$B\rho^4\,(\rho^6 + 75\rho^4 - 141\rho^2 + 1) = - 7A\,(\rho^6 - 333\rho^4 + 1947\rho^2 + 49)$$
$$- 56\,\big(3A^2 + 2\rho^2\,(14\rho^4 - 119\rho^2 + 1)\big)(\rho^2 + 7),$$

which is the value of B, expressed rationally in terms of ρ, A; it will be observed that B is obtained as a quadric function of A, which is the proper form.

Writing $\rho = - 1$, we have $A = - 15$, $B = - 287$, $t_1 = - 64$, $q_1 = - 163$, and the equation is

$$287.64 = 105.1664 - 56.349.8, = 174720 - 156352, = 18368,$$

which is right.

65. Writing for B, A their values $= - \dfrac{56}{\rho}\,\beta - \dfrac{343}{\rho^2}$, and $8\rho\alpha - 7\rho^2$, we have

$$\rho^4\Big(- \frac{56}{\rho}\,\beta - \frac{343}{\rho^2}\Big)\,t_1 = (- 56\rho\alpha + 49\rho^2)(\rho^6 - 333\rho^4 + 1947\rho^2 + 49)$$
$$- 56\,(192\rho^2\alpha^2 - 336\rho^3\alpha + 147\rho^4 + 2\rho^2 q_1)(\rho^2 + 7);$$

that is, $\qquad - 56\rho^3\beta t_1 = - 56.192\rho^2\,(\rho^2 + 7)\,\alpha^2$
$$- 56\rho\alpha\,(\rho^6 - 333\rho^4 + 1947\rho^2 + 49)$$
$$+ 56.336\rho^3\alpha\,(\rho^2 + 7)$$
$$+ 49\rho^2\,(\rho^6 - 333\rho^4 + 1947\rho^2 + 49)$$
$$- 56\,\big(147\rho^4 + 2\rho^2\,(14\rho^4 - 119\rho^2 - 58)\big)(\rho^2 + 7)$$
$$+ 343\rho^2\,(\rho^6 + 75\rho^4 - 141\rho^2 + 1),$$

where the fourth and sixth lines unite into a term divisible by 56, viz. omitting in the first instance a factor 49, the lines are

$$\rho^8 - 333\rho^6 + 1947\rho^4 + 49\rho^2$$

and $\qquad\qquad\qquad 7\rho^6 + 525\rho^6 - 987\rho^4 + 7\rho^2,$

which together are $\quad = 8\rho^8 + 192\rho^6 + 960\rho^4 + 56\rho^2,$

and hence, restoring the factor 49, the lines are

$$= 392\,(\rho^8 + 24\rho^6 + 120\rho^4 + 7\rho^2),$$

and the formula now easily becomes

$$\begin{aligned}
\rho^2 \beta t_1 =\ & 192\rho\,(\rho^2 + 7)\,\alpha^2 \\
& + (\rho^6 - 669\rho^4 - 405\rho^2 + 49)\,\alpha \\
& + \rho\,(21\rho^6 - 63\rho^4 - 1593\rho^2 - 861),
\end{aligned}$$

where the last line is

$$= \rho\,(\rho^2 + 7)(21\rho^4 - 210\rho^2 - 123).$$

66. Hence, finally, substituting for t_1 its value, we have

$$\begin{aligned}
\beta\rho^2\,(\rho^6 + 75\rho^4 - 141\rho^2 + 1) =\ & 3\rho\,(\rho^2 + 7)(64\alpha^2 + 7\rho^4 - 70\rho^2 - 41) \\
& + \alpha\,(\rho^6 - 669\rho^4 - 405\rho^2 + 49),
\end{aligned}$$

which is the expression for β as a rational function of ρ, α.

Here $\rho = 1$, $\alpha = -1$, $\beta = -1$ give $64 = -960 + 1024$, which is right, and again $\rho = -7$, $\alpha = -1$, $\beta = -1$ give

$$\begin{aligned}
-49\,(117649 + 180075 - 6909 + 1) =\ & -21.56\,(64 + 16807 - 3430 - 41) \\
& - (117649 - 1606269 - 19845 + 49);
\end{aligned}$$

that is $\qquad -49.290816 = -1176.13400 + 1508416,$

or $\qquad -14249984\ \ = -15758400 + 1508416,$ which is right.

The $\alpha\beta$-Differential Equation. Art. No. 67.

67. We have, No. 10, $\qquad \dfrac{d\beta}{\beta^2 - 1} = \dfrac{\rho^2}{7}\,\dfrac{d\alpha}{\alpha^2 - 1},$

and it should of course be possible to verify this equation by means of the $\rho\alpha$-equation and the value just obtained for β. But the expression for $\dfrac{d\rho}{d\alpha}$ given by the $\rho\alpha$-equation is of so complicated a form that I do not see in what way the verification will come out, and I have not attempted to effect it.

The Coefficients A_1 and A_2. Art. Nos. 68 to 71.

68. These are given by the formulæ No. 47, viz. we have

$$A_1 = \frac{1}{\rho}\,7\,(\alpha^2 - 1)\,\frac{d\rho}{d\alpha} - \frac{7}{2}\,\alpha\ + \frac{1}{2}\,\beta\rho^2,$$

$$A_2 = \quad 7\,(\alpha^2 - 1)\,\frac{d\rho}{d\alpha} - \frac{19}{6}\,\alpha\rho + \frac{1}{6}\,\beta\rho^3,$$

where $\dfrac{d\rho}{d\alpha}$ and β have each of them to be expressed in terms of ρ, α; we have thus A_1 and A_2 each of them expressible rationally in terms of ρ, α; but I have not attempted to effect the substitutions.

69. The five equations of No. 42, merely collecting the terms, are

$$12A_2 - 6A_1^2 - 8\alpha A_1 + \rho^4 - 7 = 0,$$
$$(-6A_1 - 32\alpha + 2\rho^3)A_2 - 2A_1^3 - 8A_1 + 30\rho = 0,$$
$$(\rho^2 - 4)A_2^2 + (-4A_1^2 - 8\alpha A_1 + 6)A_2 - 5A_1^2 + (2\rho^3 + 4\rho)A_1 - 72\alpha\rho = 0,$$
$$- 2A_1A_2^2 + \{(2\rho^2 - 4)A_1 - 6\rho\}A_2 - 4\rho A_1^2 - 32\rho\alpha A_1 + 2\rho^3 + 28\rho = 0,$$
$$- 3A_2^2 + (-4\rho A_1 + 2\rho^2 - 8\alpha\rho)A_2 + \rho^2 A_1^2 + 10\rho A_1 - 6\rho^2 = 0,$$

which would of course be all of them satisfied by the values of A_1, A_2 as rational functions of ρ, α, viz the substitution of these values in any one of the equations would give a function of ρ, α containing as a factor the expression on the left-hand side of the $\rho\alpha$-equation.

70. Or again, the equations should determine A_1 and A_2 as rational functions of ρ, α, but there is no obvious way of finding such values in a simple form. We of course have $\qquad 12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7,$

and using this value to eliminate A_2 from the remaining equations we find the following four equations:

$$A_1^3 \cdot 30 + A_1^2(120\alpha - 6\rho^3) + A_1\{128\alpha^2 - 8\rho^3\alpha - 3\rho^4 + 69\}$$
$$+ \alpha(-16\rho^4 + 112) + \rho^7 - 7\rho^3 - 180\rho = 0,$$
$$A_1^4(36\rho^2 - 432) + A_1^3\alpha(96\rho^2 - 1344)$$
$$+ A_1^2\{\alpha^2(64\rho^2 - 1024) - 12\rho^6 + 48\rho^4 + 84\rho^2 - 624\}$$
$$+ A_1\{\alpha(-16\rho^6 + 160\rho^4 + 112\rho^2 - 544) + 288\rho^3 + 576\rho\}$$
$$+ \{-10368\alpha\rho + \rho^{10} - 4\rho^8 - 14\rho^6 - 16\rho^4 + 49\rho^2 - 308\} = 0,$$
$$A_1^5 \cdot 36 + A_1^4 \cdot 96\alpha + A_1^3\{64\alpha^2 - 12\rho^4 - 72\rho^2 + 208\}$$
$$+ A_1^2\{\alpha(-16\rho^4 - 96\rho^2 + 304) + 504\rho\}$$
$$+ A_1\{\alpha \cdot 2592\rho + \rho^8 - 12\rho^6 + 10\rho^4 + 84\rho^2 - 119\}$$
$$+ 36\rho^5 - 144\rho^3 - 2268\rho = 0,$$
$$A_1^4 \cdot 36 + A_1^3(96\alpha + 96\rho) + A_1^2\{64\alpha^2 + 320\alpha\rho - 12\rho^4 - 96\rho^2 + 84\}$$
$$+ A_1\{256\alpha^2\rho + \alpha(-80\rho^4 + 112) - 16\rho^5 - 368\rho\}$$
$$+ \{\alpha(-32\rho^5 + 224\rho) + \rho^8 + 8\rho^6 - 14\rho^4 + 232\rho^2 + 49\} = 0,$$

and we could from these equations obtain various rational expressions for A_1 and its powers, but these would apparently be of degrees far too high in ρ and α.

71. It is to be remarked that for $\rho = 1$, $\alpha = -1$, the values of A_1, A_2 are $A_1 = A_2 = 3$, viz. these belong to the solution

$$y = \frac{x(1 + 3x^2 + 3x^4 + x^6)}{1 + 3x^2 + 3x^4 + x^6}, \; = x, \; \text{of} \; \frac{dy}{1 + y^2} = \frac{dx}{1 + x^2};$$

and that for $\rho = -7$, $\alpha = -1$, the values are $A_1 = -21$, $A_2 = 35$, viz. these belong to the solution

$$y = \frac{-7x + 35x^3 - 21x^5 + x^7}{1 - 21x^2 + 35x^4 - 7x^6} \; \text{of} \; \frac{dy}{1 + y^2} = \frac{-7dx}{1 + x^2}.$$

For example, the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ becomes, for the first set of values, $36 = 54 - 24 - 1 + 7$, and for the second set of values, $420 = 2646 + 168 - 2401 + 7$, which are each of them right.

New Form of the Seventhic Transformation.   Art. Nos. 72 to 83.

72. For the quartic function $1 - 2\alpha x^2 + x^4$, the coefficients a, b, c, d, e are $= 1, 0, -\dfrac{1}{3}\alpha, 0, 1$, and hence the invariants I, J and the discriminant Δ are

$$I = 1 + \frac{1}{3}\alpha^2, = \frac{1}{3}(\alpha^2 + 3),$$

$$J = -\frac{1}{3}\alpha + \frac{1}{27}\alpha^3, = \frac{1}{27}(\alpha^3 - 9\alpha),$$

$$\Delta = I^3 - 27J^2, = \frac{1}{27}\{(\alpha^2 + 3)^3 - (\alpha^3 - 9\alpha)^2\}, = (\alpha^2 - 1)^2, \; \text{whence} \; \sqrt[12]{\Delta} = \sqrt[6]{\alpha^2 - 1}.$$

This being so, then assuming $\rho = p \dfrac{\sqrt[6]{\alpha^2 - 1}}{\sqrt[6]{\beta^2 - 1}},$

the differential equation

$$\frac{dy}{\sqrt{1 - 2\beta y^2 + y^4}} = \frac{\rho dx}{\sqrt{1 - 2\alpha x^2 + x^4}}$$

becomes

$$\frac{\sqrt[6]{\beta^2 - 1}\, dy}{\sqrt{1 - 2\beta y^2 + y^4}} = \frac{p\sqrt[6]{\alpha^2 - 1}\, dx}{\sqrt{1 - 2\alpha x^2 + x^4}},$$

viz. this is, for the radicals $\sqrt{1 - 2\alpha x^2 + x^4}$ and $\sqrt{1 - 2\beta y^2 + y^4}$, the form considered by Klein in the paper "Ueber die Transformation der Elliptischen Fonctionen und die Auflösung der Gleichungen fünften Grades," Math. Ann., t. XIV (1879), pp. 111–172. I notice that there is some error as to a factor 7, and that p is equal to the z of p. 148, not as might appear $= \dfrac{1}{7}z$.

73. The modular equation presents itself in the form given p. 143, viz. this is

$$\mathbf{J} : \mathbf{J} - 1 : 1 = (\tau^2 + 13\tau + 49)(\tau^2 + 5\tau + 1)^3 : (\tau^4 + 14\tau^3 + 63\tau^2 + 70\tau - 7)^2 : 1728\tau,$$

with the like relation in $\mathbf{J}'$, τ', and then $\tau\tau' = 49$. We have thus $\mathbf{J}$, $\mathbf{J}'$ each given as a function of τ, and thence by elimination of τ we have the modular equation as a relation between the absolute invariants $\mathbf{J}$, $\mathbf{J}'$. But $\tau = p^2$, and for the form $1 - 2\alpha x^2 + x^4$, as appears above, we have

$$\mathbf{J} - 1, = \frac{27J^2}{\Delta}; = \frac{\frac{1}{27}(\alpha^3 - 9\alpha)^2}{(\alpha^2 - 1)^2};$$

hence Klein's equation

$$\mathbf{J} - 1 = \frac{(\tau^4 + 14\tau^3 + 63\tau^2 + 70\tau - 7)^2}{1728\tau}$$

becomes

$$\frac{\alpha^3 - 9\alpha}{\alpha^2 - 1} = \frac{p^8 + 14p^6 + 63p^4 + 70p^2 - 7}{8p};$$

or say

$$p^8 + 14p^6 + 63p^4 + 70p^2 - 8\left(\frac{\alpha^3 - 9\alpha}{\alpha^2 - 1}\right)p - 7 = 0,$$

(which is the equation p. 148 with p for z), viz. this is the $p\alpha$-equation connecting α with the new multiplier p. It will be observed that it is of the degree 8 in p, and the degree 3 in α, viz. it resembles herein the foregoing $\rho\alpha$-equation, but the form is very much more simple, inasmuch as the α enters into a single coefficient only. The equation may also be written

$$(p^4 + 5p^2 + 1)^3(p^4 + 13p^2 + 49) - 64\frac{(\alpha^2 + 3)^3}{(\alpha^2 - 1)^2}p^2 = 0.$$

74. Using for shortness a single letter m to denote the value $i\sqrt{3}$, we have

$$\frac{\alpha^3 - 9\alpha + 3m(\alpha^2 - 1)}{\alpha^3 - 9\alpha - 3m(\alpha^2 - 1)} = \frac{p^8 + 14p^6 + 63p^4 + 70p^2 + 24mp - 7}{p^8 + 14p^6 + 63p^4 + 70p^2 - 24mp - 7};$$

that is

$$\left(\frac{\alpha + m}{\alpha - m}\right) = \frac{(p^2 - mp + 1)^3(p^2 + 3mp - 7)}{(p^2 + mp + 1)^3(p^2 - 3mp - 7)},$$

or say

$$\frac{\alpha + m}{\alpha - m} = \frac{p^2 - mp + 1}{p^2 + mp + 1}\sqrt[3]{\frac{p^2 + 3mp - 7}{p^2 - 3mp - 7}},$$

which is another form of the $p\alpha$-equation.

75. We had $\tau = p^2$, and similarly writing $\tau' = q^2$, then $\tau\tau' = 49 = p^2q^2$; it must be assumed that $pq = -7$; β is then the same function of q which α is of p, viz. we have

$$\frac{\beta + m}{\beta - m} = \frac{q^2 - mq + 1}{q^2 + mq + 1}\sqrt[3]{\frac{q^2 + 3mq - 7}{q^3 - 3mq - 7}}.$$

These equations in α and β contain the same cubic radical, viz. we have

$$q^2 + 3mq - 7, = \frac{49}{p^2} - \frac{21m}{p} - 7, = -\frac{7}{p^2}(p^2 + 3mp - 7),$$

and similarly

$$q^2 - 3mq - 7 \qquad\qquad = -\frac{7}{p^2}(p^2 - 3mp - 7).$$

Moreover

$$q^2 - mq + 1, = \frac{49}{p^2} + \frac{7m}{p} + 1, = \frac{1}{p^2}(p^2 + 7mp + 49),$$

and similarly

$$q^2 + mq + 1 \qquad\qquad = \frac{1}{p^2}(p^2 - 7mp + 49),$$

and we thus obtain

$$\frac{\beta + m}{\beta - m} = \frac{p^2 + 7mp + 49}{p^2 - 7mp + 49}\sqrt[3]{\frac{p^2 + 3mp - 7}{p^2 - 3mp - 7}},$$

whence, eliminating the cubic radical,

$$\frac{\beta + m}{\beta - m} = \frac{p^2 + 7mp + 49}{p^2 - 7mp + 49}\frac{p^2 + mp + 1}{p^2 - mp + 1}\frac{\alpha + m}{\alpha - m},$$

viz. this gives β as a rational function of α, p. We in fact have

$$\beta = \frac{\alpha(p^4 + 29p^2 + 49) - 24p(p^2 + 7)}{\alpha.8p(p^2 + 7) + (p^4 + 29p^2 + 49)}.$$

76. The differential relation $\dfrac{d\beta}{\beta^2 - 1} = \dfrac{\rho^2}{7}\dfrac{d\alpha}{\alpha^2 - 1}$, substituting therein for ρ its value, becomes

$$\frac{d\beta}{(\beta^2 - 1)^{\frac{2}{3}}} = \frac{p^2}{7}\frac{d\alpha}{(\alpha^2 - 1)^{\frac{2}{3}}}.$$

But, from the expression for $\dfrac{\alpha + m}{\alpha - m}$, we obtain

$$d\alpha\left(\frac{1}{\alpha + m} - \frac{1}{\alpha - m}\right)$$

$$= dp\left\{\left(\frac{2p - m}{p^2 - mp + 1} - \frac{2p + m}{p^2 + mp + 1}\right) + \frac{1}{3}\left(\frac{2p + 3m}{p^2 + 3mp - 7} - \frac{2p - 3m}{p^2 - 3mp - 7}\right)\right\},$$

or, omitting from each side a factor $- 2m$,

$$\frac{d\alpha}{\alpha^2 + 3} = dp\left(\frac{-p^2 + 1}{p^4 + 5p^2 + 1} + \frac{p^2 + 7}{p^4 + 13p^2 + 49}\right) = \frac{56dp}{(p^4 + 5p^2 + 1)(p^4 + 13p^2 + 49)}.$$

But we have, No. 73,

$$\frac{\alpha^2 + 3}{(\alpha^2 - 1)^{\frac{2}{3}}} \qquad\qquad = \frac{(p^4 + 5p^2 + 1)(p^4 + 13p^2 + 49)^{\frac{1}{3}}}{4p^{\frac{2}{3}}},$$

and thence

$$\frac{d\alpha}{(\alpha^2 - 1)^{\frac{2}{3}}} = \frac{14 dp}{p^{\frac{2}{3}}(p^4 + 13p^2 + 49)^{\frac{2}{3}}},$$

and similarly

$$\frac{d\beta}{(\beta^2 - 1)^{\frac{2}{3}}} = \frac{14 dq}{q^{\frac{2}{3}}(q^4 + 13q^2 + 49)^{\frac{2}{3}}}.$$

The equation $q = -\dfrac{7}{p}$ gives

$$dq = \frac{7 dp}{p^2}, \quad q^{\frac{2}{3}}(q^4 + 13q^2 + 49)^{\frac{2}{3}} = 49 p^{-\frac{10}{3}}(p^4 + 13p^2 + 49)^{\frac{2}{3}},$$

and we thence have

$$\frac{d\beta}{(\beta^2 - 1)^{\frac{2}{3}}} = \frac{2 p^{\frac{4}{3}} dp}{(p^4 + 13p^2 + 49)^{\frac{2}{3}}}, \quad = \frac{p^2}{7} \frac{d\alpha}{(\alpha^2 - 1)^{\frac{2}{3}}},$$

the required relation.

77. From the value of ρ we have

$$\frac{d\rho}{\rho} = \frac{dp}{p} + \frac{\frac{1}{3}\alpha d\alpha}{\alpha^2 - 1} - \frac{\frac{1}{3}\beta d\beta}{\beta^2 - 1},$$

which, substituting for $d\beta$ its value, becomes

$$= \frac{dp}{p} + \frac{\frac{1}{3} d\alpha}{(\alpha^2 - 1)^{\frac{2}{3}}} \left\{ \frac{\alpha}{(\alpha^2 - 1)^{\frac{1}{3}}} - \frac{\beta}{(\beta^2 - 1)^{\frac{1}{3}}} \frac{p^2}{7} \right\},$$

or say

$$\frac{1}{\rho}\frac{d\rho}{d\alpha} = \frac{1}{p}\frac{dp}{d\alpha} + \frac{\frac{1}{3}}{(\alpha^2 - 1)^{\frac{2}{3}}} \left\{ \frac{\alpha}{(\alpha^2 - 1)^{\frac{1}{3}}} - \frac{\beta}{(\beta^2 - 1)^{\frac{1}{3}}} \frac{p^2}{7} \right\},$$

which, however, is more conveniently written

$$\frac{1}{\rho}\frac{d\rho}{d\alpha} = \frac{1}{p}\frac{dp}{d\alpha} + \frac{\frac{1}{3}}{\alpha^2 - 1}(\alpha - \beta\rho^2);$$

and then substituting in the formulæ for A_1, A_2 we find

$$A_1 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \frac{7}{6}\alpha + \frac{1}{6}\beta\rho^2,$$

$$\frac{1}{\rho}A_2 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \frac{5}{6}\alpha - \frac{1}{6}\beta\rho^2,$$

$\left(\text{expressions which give, as they should do, } A_2 - \rho A_1 = \frac{1}{3}(\alpha\rho - \beta\rho^3)\right)$. In

these last formulæ ρ is to be regarded as standing for its value, $= p\,\dfrac{\sqrt[6]{\alpha^2 - 1}}{\sqrt[6]{\beta^2 - 1}}$.

78. To further reduce these values, consider the expression of β given No. 75. If for a moment we represent this by

$$\beta = \frac{F\alpha - 3G}{G\alpha + F}, \text{ where } F = p^4 + 29p^2 + 49, \quad G = 8p\,(p^2 + 7),$$

then we have

$$\beta^2 - 1 = \frac{(F^2 - G^2)\,\alpha^2 - 8FG\alpha + 9G^2 - F^2}{(G\alpha + F)^2},$$

or, multiplying the numerator and denominator each by $G\alpha + F$, so as to make the denominator a perfect cube, the numerator becomes

$$G\,(F^2 - G^2)(\alpha^3 - 9\alpha) + F\,(F^2 - 9G^2)(\alpha^2 - 1),$$

and putting for the factor G of the first term its value $= 8p\,(p^2 + 7)$, we thus obtain

$$\frac{\beta^2 - 1}{\alpha^2 - 1} = \frac{(F^2 - G^2)(p^2 + 7)\,8p\left(\dfrac{\alpha^3 - 9\alpha}{\alpha^2 - 1}\right) + F(F^2 - 9G^2)}{(G\alpha + F)^3},$$

viz. in virtue of the $p\alpha$-equation, this is

$$\frac{\beta^2 - 1}{\alpha^2 - 1} = \frac{(F^2 - G^2)(p^2 + 7)(p^8 + 14p^6 + 63p^4 + 70p^2 - 7) + F(F^2 - 9G^2)}{(G\alpha + F)^3}.$$

This numerator is $= (p^4 + 5p^2 + 1)^3 p^6$; in fact we have

$$(F^2 - G^2)(p^2 + 7) = p^{10} + p^8 + \quad p^6 + \quad 7p^4 + \quad 343p^2 + 16807,$$
$$F^2 - 9G^2 \qquad = \qquad p^8 - 518p^6 - 7125p^4 - 25382p^2 + \quad 2401,$$

and thence forming the two terms of the numerator and adding them together—for shortness I write down only the coefficients—we have

1	15	78	154	567	22113	257390	1082802	1174089	−117649
		1	−489	−22098	−257389	−1082802	−1174089	117649

$$= 1 \quad 15 \quad 78 \quad 155 \quad 78 \quad 15 \quad 1 \quad 0 \quad 0 \quad 0$$

viz. these are the coefficients of $(p^4 + 5p^2 + 1)^3 p^6$. Hence

$$\frac{\beta^2 - 1}{\alpha^2 - 1} = \frac{(p^4 + 5p^2 + 1)^3 p^6}{(G\alpha + F)^3};$$

or, extracting the cube root, and for G, F substituting their values,

$$\frac{\sqrt[3]{\beta^2 - 1}}{\sqrt[3]{\alpha^2 - 1}} = \frac{(p^4 + 5p^2 + 1)\,p^2}{8p\,(p^2 + 7)\,\alpha + p^4 + 29p^2 + 49},$$

and thence also

$$o^2 = \frac{8p\,(p^2 + 7)\,\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1},$$

viz. we have thus ρ^2 expressed as a rational function of p, α.

79. It will presently appear that ρ is in fact expressible as a rational function of p, α, but I am unable to obtain this expression in a simple form. Admitting that ρ is thus expressible, a direct process for obtaining the expression is as follows. Writing

$$\xi = \frac{8p\,(p^2 + 7)\,\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1} \quad (= \rho^2),$$

and by means hereof introducing ξ in place of α into the equation

$$p^8 + 14p^6 + 63p^4 + 70p^2 - 8p\,\frac{\alpha^3 - 9\alpha}{\alpha^2 - 1} - 7 = 0,$$

we have for ξ a cubic equation,

$$a\xi^3 + b\xi^2 + c\xi + d = 0,$$

where the coefficients a, b, c, d are given rational functions of p. This equation may be written

$$a\xi\,(\xi + \vartheta)^2 + b'\xi^2 + c'\xi + d = 0,$$

where $b' = b - 2a\vartheta$, $c' = c - a\vartheta^2$; and the last three terms will be a square if only $c'^2 - 4b'd = 0$; that is, if

$$(a\vartheta^2 - c)^2 + 4d\,(2a\vartheta - b) = 0,$$

a biquadratic equation in ϑ which (ρ being expressible as above) must have one of its roots $=$ a rational function of p. Calling this ϑ, we then have

$$a\xi\,(\xi + \vartheta)^2 + \frac{1}{b'}\left(b'\xi + \frac{1}{2}\,c'\right)^2 = 0, \quad \text{or say } a\rho^2\,(\xi + \vartheta)^2 + \frac{1}{b'}\left(b'\xi + \frac{1}{2}\,c'\right)^2 = 0,$$

hence

$$\rho = \sqrt{\frac{-1}{ab'}} \cdot \frac{b'\xi + \frac{1}{2}\,c'}{\xi + \vartheta},$$

where ξ denotes a linear function of α as above; the quadric radical will have a rational value, and the form of the equation thus is

$$\rho = \frac{A\alpha + B}{C\alpha + D},$$

where A, B, C, D are rational and integral functions of p. But I am not able to carry out the process.

80. As shown, No. 78, we have

$$\rho^2 = \frac{8p\,(p^2 + 7)\,\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1}.$$

Multiplying by the value of β, *ante* No. 75, we find

$$\beta\rho^2 = \frac{(p^4 + 29p^2 + 49)\,\alpha - 24p\,(p^2 + 7)}{p^4 + 5p^2 + 1}.$$

and we can hence find A_1 and A_2 by the formulæ

$$A_1 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \frac{7}{6}\alpha + \frac{1}{6}\beta\rho^2,$$

$$\frac{1}{\rho}A_2 = 7(\alpha^2 - 1)\frac{1}{p}\frac{dp}{d\alpha} - \frac{5}{6}\alpha - \frac{1}{6}\beta\rho^2,$$

or, for the second of these we may write

$$\frac{1}{\rho}A_2 = A_1 + \frac{1}{3}(\alpha - \beta\rho^2).$$

But in a different point of view, regarding only ρ^2, but not ρ, as a given function of p, α, we must to these equations join the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$, *ante* No. 69, and we have thus equations for the determination of A_1, A_2, and ρ.

81. We have

$$A_1 = \frac{(p^4 + 5p^2 + 1)(p^4 + 13p^2 + 49)}{8p} \cdot \frac{\alpha^2 - 1}{\alpha^2 + 3}$$

$$- \frac{7}{6}\alpha + \frac{\alpha(p^4 + 29p^2 + 49) - 24p(p^2 + 7)}{6(p^4 + 5p^2 + 1)},$$

where the second line is

$$= \frac{\alpha(-p^4 - p^2 + 7) - 4p(p^2 + 7)}{p^4 + 5p^2 + 1}.$$

Uniting the two terms, we have a denominator $8p(p^4 + 5p^2 + 1)$, and in the numerator a term $8p\alpha^3$ which may be got rid of by means of the $p\alpha$-equation; the numerator thus becomes

$$= 96p(-p^4 - p^2 + 7) - 128p^2(p^2 + 7)\alpha$$
$$+ (\alpha^2 - 1)\{(-p^4 - p^2 + 7)(p^8 + 14p^6 + 63p^4 + 70p^2 - 7)\}$$
$$+ (p^4 + 5p^2 + 1)^2(p^4 + 13p^2 + 49) - 32p^2(p^2 + 7),$$

where the whole divides by $8p$, and we finally obtain

$$A_1 = \frac{12(-p^4 - p^2 + 7) - 16p(p^2 + 7)\alpha + (\alpha^2 - 1)p(p^8 + 17p^6 + 102p^4 + 225p^2 + 97)}{(\alpha^2 + 3)(p^4 + 5p^2 + 1)}.$$

Proceeding to calculate the value of $A_1 + \frac{1}{3}(\alpha - \beta\rho^2)$, we then have

$$\frac{1}{3}(\alpha - \beta\rho^2) = \frac{-8(p^2 + 2)\alpha + 8p(p^2 + 7)}{p^4 + 5p^2 + 1}.$$

Multiplying the numerator and denominator by $\alpha^2 + 3$, we have in the numerator

a term in $8a^3$ which may be got rid of by means of the pa-equation; the numerator thus becomes

$$12(-p^4 - 9p^2 - 9) + 16p(p^2 + 7) + (a^2 - 1)p\{p^8 + 17p^6 + 102p^4 + 225p^2 + 97\}$$
$$- \frac{p^2 + 2}{p^2}(p^8 + 14p^6 + 63p^4 + 70p^2 - 7) + 8(p^2 + 7),$$

and we finally obtain

$$\frac{1}{\rho}A_2 = \frac{12(-p^4 - 9p^2 - 9) + 16p(p^2 + 7) + (a^2 - 1)p^{-1}(p^8 + 11p^6 + 37p^4 + 20p^2 + 2)}{(a^2 + 3)(p^4 + 5p^2 + 1)}.$$

82. The expressions obtained above for ρ^2, A_1, A_2 are of the form

$$\rho^2 = \frac{M + Na}{S}, \quad A_1 = \frac{P_1 + Q_1 a + R_1 a^2}{S(a^2 + 3)}, \quad \frac{1}{\rho}A_2 = \frac{P_2 + Q_2 a + R_2 a^2}{S(a^2 + 3)},$$

where

$M = p^4 + 29p^2 + 49;$ $N = 8p(p^2 + 7);$ $S = p^4 + 5p^2 + 1,$
$P_1 = 12(-p^4 - p^2 + 7) - p(p^8 + 17p^6 + 102p^4 + 225p^2 + 97),$ $Q_1 = -16p(p^2 + 7),$
$R_1 = \qquad\qquad p(p^8 + 17p^6 + 102p^4 + 225p^2 + 97);$
$P_2 = 12(-p^4 - 9p^2 - 9) - p^{-1}(p^8 + 11p^6 + 37p^4 + 20p^2 + 2),$ $Q_2 = \quad 16p(p^2 + 7),$
$R_2 = \qquad\qquad p^{-1}(p^8 + 11p^6 + 37p^4 + 20p^2 + 2);$

and substituting these values in the foregoing equation

$$12A_2 = 6A_1^2 + 8aA_1 - \rho^4 + 7,$$

we obtain

$$12\rho\left\{\frac{P_2 + Q_2 a + R_2 a^2}{S(a^2 + 3)}\right\} = \left\{\frac{6(P_1 + Q_1 a + R_1 a^2)^2}{S^2(a^2 + 3)^2} + 8a\frac{P_1 + Q_1 a + R_1 a^2}{S(a^2 + 3)} - \frac{(M + Na)^2}{S^2} + 7\right\};$$

that is,

$$\rho = \frac{1}{12(P_2 + Q_2 a + R_2 a^2)^2(3 + a^2)S}\{6(P_1 + Q_1 a + R_1 a^2)^2 + 8aS(3 + a^2)(P_1 + Q_1 a + R_1 a^2)$$
$$- (M + Na)^2(3 + a^2)^2 + 7S^2(3 + a^2)^2\},$$

which, by means of the pa-equation

$$p^8 + 14p^6 + 63p^4 + 70p^2 - \left(\frac{a^3 - 9a}{a^2 - 1}\right)8p - 7 = 0,$$

should be reducible to the form

$$\rho = Aa + Ba + C, \quad \text{or} \quad \rho = \frac{Aa + B}{Ca + D};$$

but I have not been able to obtain in either of these forms a simple expression of ρ as a function of p, a. Supposing it obtained, the ρa-equation, *ante* No. 51, would of course be thereby transformable into the foregoing pa-equation. And considering p as an auxiliary parameter thus introduced into the formulæ in place of ρ, then β and the coefficients A_1, A_2 are, by what precedes, expressed in

terms of p, α; that is, in effect in terms of ρ, α, and we thus have the formulæ of transformation for the $\rho\alpha\beta$-form.

83. There exists a remarkably simple particular case. Write for convenience $\theta = \sqrt{7}$; the $p\alpha$-equation is satisfied by the values $p = -\theta$, $\alpha = -\dfrac{3}{8}\theta$.

In fact, these values give $8p\alpha = 3\theta^2$, $= 21$, $\dfrac{\alpha^2 - 9}{\alpha^2 - 1} = \left(\dfrac{63}{64} - 9\right) \div \left(\dfrac{63}{64} - 1\right)$, $= 513$; the term in α is thus 21.513, $= 10773$; but, assuming $p^2 = 7$, we have $p^8 + 14p^6 + 63p^4 + 70p^2 - 7 = 2401 + 4802 + 3087 + 490 - 7$, $= 10773$, and the equation is thus satisfied. And these values, $p = -\theta$, $\alpha = -\dfrac{3}{8}\theta$, give $\rho^2 = 7$, $\beta = \dfrac{3}{8}\theta$, $A_1 = 2\theta$, $A_2 = \rho\theta$; the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ thus becomes $12\rho\theta = 168 - 42 - 49 + 7$, $= 84$; that is, $\rho\theta = 7$, $= \theta^2$, or $\rho = \theta$ ($= -p$). We have $\alpha^2 - 1$, $= \beta^2 - 1$, $= -\dfrac{1}{64}$; but from the equation $\rho = p\dfrac{\sqrt[6]{\alpha^2 - 1}}{\sqrt[6]{\beta^2 - 1}}$, it appears that the sixth roots must be equal with opposite signs, say $\sqrt[6]{\alpha^2 - 1} = \dfrac{i}{2}$, $\sqrt[6]{\beta^2 - 1} = \dfrac{-i}{2}$. Retaining θ to stand for its value $= \sqrt{7}$, the differential equation is

$$\frac{dy}{\sqrt{1 - \dfrac{3}{4}\theta y^2 + y^4}} = \frac{\theta dx}{\sqrt{1 + \dfrac{3}{4}\theta x^2 + x^4}},$$

satisfied by
$$y = \frac{x(\theta + 7x^2 + 2\theta x^4 + x^6)}{1 + 2\theta x^2 + 7x^4 + \theta x^6}.$$

It may be remarked that the quartic functions of y and x resolved into their linear factors are

$$\left\{y + \frac{3i + \theta}{2\sqrt{2}(1 + i)}\right\}\left\{y + \frac{3i - \theta}{2\sqrt{2}(1 + i)}\right\}\left\{y + \frac{-3i + \theta}{2\sqrt{2}(1 - i)}\right\}\left\{y + \frac{-3i - \theta}{2\sqrt{2}(1 - i)}\right\}$$

and

$$\left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x + \frac{3 + i\theta}{2\sqrt{2}(1 + i)}\right\}\left\{x + \frac{3 - i\theta}{2\sqrt{2}(1 - i)}\right\}\left\{x + \frac{3 + i\theta}{2\sqrt{2}(1 - i)}\right\},$$

and that for the first of the y-factors, substituting for y its value, we have

$$x^7 + 2\theta x^5 + 7x^3 + \theta x + \frac{3i + \theta}{2\sqrt{2}(1 + i)}(\theta x^6 + 7x^4 + 2\theta x^2 + 1)$$

$$= \left(x + \frac{3 - i\theta}{2\sqrt{2}(1 + i)}\right)\left\{x^3 + \frac{1 + i\theta}{\sqrt{2}(1 + i)}x^2 + \frac{1}{2}(i + \theta)x + \frac{1 + i}{\sqrt{2}}\right\}^2,$$

with like expressions for the other y-factors respectively.

Brioschi's Transformation Theory.   Art. No. 84.

84. M. Brioschi has kindly referred me to two papers by him, "Sur une Formule de Transformation des Fonctions Elliptiques," Comptes Rendus, t. 79 (1874), pp. 1065–1069, and ibid. t. 80 (1875), pp. 261–264. They relate to the form

$$\frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} = \frac{dy}{\sqrt{4y^3 - G_2 y - G_3}},$$

with a formula of transformation

$$y = \frac{U}{T^2}, \quad T = x^\nu + a_1 x^{\nu-1} + a_2 x^{\nu-2} \ldots + a_\nu \left\{ \nu = \frac{1}{2}(n-1) \right\}$$

$$U = x^n + a_1 x^{n-2} + a_2 x^{n-3} \ldots + a_\nu.$$

The general theory for any value of n is developed to a considerable extent, and it would without doubt give very interesting results for the case $n = 7$; but the formulæ are only completely worked out for the preceding two cases $n = 3$ and $n = 5$. For these cases the formulæ are as follows:

Cubic transformation: $n = 3$,

$$y = \frac{x^3 + a_1 x^2 + a_2 x + a_3}{(x + a_1)^2}.$$

Corresponding to the modular equation we have

$$a_1^4 - \frac{1}{2} g_2 a_1^2 + g_3 a_1 - \frac{1}{48} g_2^2 = 0,$$

and then

$$G_2 - 9g_2 = 6\left(20a_1^2 - 3g_2\right), \quad G_3 + 27g_3 = -14\left(20a_1^2 - 3g_2\right)a_1,$$

whence also

$$a_1 = -\frac{3}{7} \frac{G_3 + 27g_3}{G_2 - 9g_2},$$

and by the general theory a_1, a_2, a_3 are given rationally in terms of a_1, g_2, g_3.

Quintic transformation: $n = 5$,

$$y = \frac{x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5}{(x^2 + a_1 x + a_2)^2}.$$

We have

$$a_1 X - 2Y = 0, \quad \left(12a_1^2 + g_2\right)X - 30a_1 Y = 0,$$

where

$$X = a_1^3 - 6a_1^2 a_2 + \frac{1}{2} g_2 a_1 - g_3,$$

$$Y = 5a_2^2 - a_1^2 a_2 + \frac{1}{2} g_2 a_2 - g_3 a_1 + \frac{1}{16} g_2^2.$$

The first of these gives

$$a_2 = \frac{1}{6a_1}\left(a_1^3 + \frac{1}{2}g_2a_1 - g_3\right),$$

and then eliminating a_2, we have, corresponding to the modular equation,

$$a_1^6 - 5g_2a_1^4 + 40g_3a_1^3 - 5g_2^2a_1^2 + 8g_2g_3a_1 - 5g_3^2 = 0.$$

We then have

$$G_2 - 25g_2 = \frac{8}{a_1}(10a_1^3 - 8g_2a_1 + 5g_3), \quad G_3 + 125g_3 = -14(10a_1^3 - 8g_2a_1 + 5g_3);$$

whence also

$$a_1 = -\frac{4}{7}\frac{G_3 + 125g_3}{G_2 - 25g_2},$$

and by the general theory $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are given rationally in terms of a_1, g_2, g_3.

These results are contained in the former of the papers above referred to; the latter contains some properties of these modular equations.